~ Bellman Function Method ~

§. Example 1: Buckley's Inequality

Setup: Working in $\mathbb{R}$, and a dyadic grid $\mathcal{D}$, and dyadic A_∞ weights, i.e. weights w such that $\langle w\rangle_I \leq C e^{\langle \log w\rangle_I}$, $\forall I \in \mathcal{D}$. To formulate the inequality, consider $A_\infty^{\mathcal{D}}(J;\kappa)$, where $\kappa \geq 1$ is fixed from now on, to be a sort of κ-ball of A_∞ weights supported on some $J \in \mathcal{D}$:

$$A_\infty^{\mathcal{D}}(J;\kappa) := \{\text{weights } w \text{ s.t. } \operatorname{supp}(w) \subset J \text{ and } \langle w\rangle_I \leq \kappa e^{\langle \log w\rangle_I};\ \forall I \in \mathcal{D}(J)\}$$

Now we may state the result (found in S.M. Buckley - "Summation conditions on weights" Michigan Math. J. 1993)

Buckley's Inequality: There is an absolute constant $c(\kappa)$ such that:

$$\boxed{\sum_{I\in\mathcal{D}(J)} |I| \left(\frac{\langle w\rangle_{I_+} - \langle w\rangle_{I_-}}{\langle w\rangle_I}\right)^2 \leq |J|\, c(\kappa)} \quad \forall\, w \in A_\infty^{d}(J;\kappa).$$

I. The Black Magic Proof:

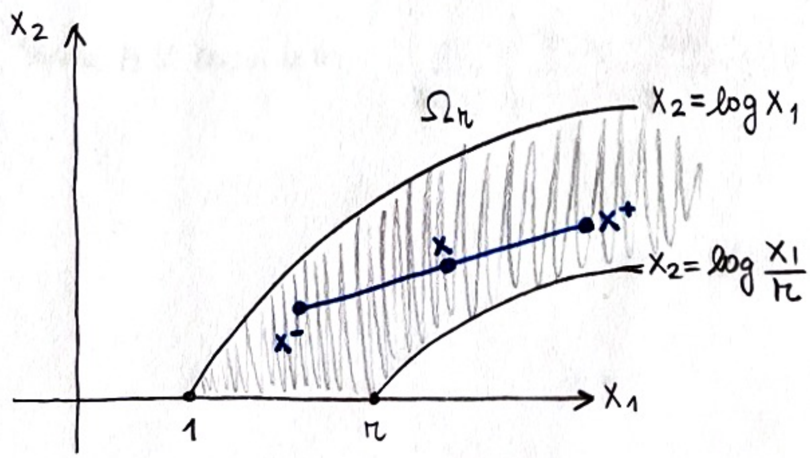
Theorem 1: Suppose there is a bounded positive function $B(x) = B(x_1,x_2)$ defined on the domain:

$$\Omega_\kappa := \{x=(x_1,x_2) : \log\frac{x_1}{\kappa} \leq x_2 \leq \log x_1;\ x_2 \geq 0\},$$

which satisfies the inequality:

$$\boxed{B(x) \geq \frac{1}{2}\left(B(x^-)+B(x^+)\right) + \left(\frac{x_1^+ - x_1^-}{x_1}\right)^2}$$

for all points $x^\pm \in \Omega_\kappa$ such that their midpoint $x := \frac{1}{2}(x^+ + x^-)$ is also in Ω_κ. Then Buckley's inequality holds.



Remark: Jensen's Inequality:

$$\langle \log w\rangle_I \leq \log \langle w\rangle_I$$

forces $\kappa \geq 1$:

$$\langle w\rangle_I \leq \kappa e^{\langle \log w\rangle_I}$$

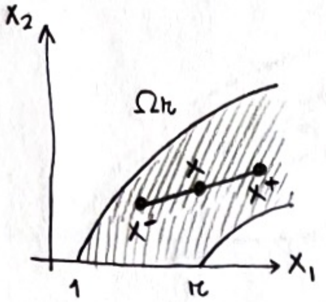
$\Rightarrow \log\langle w\rangle_I \leq \log\kappa + \langle \log w\rangle_I$

$\Rightarrow \log\kappa \geq \log\langle w\rangle_I - \langle\log w\rangle_I \geq 0$

$\Rightarrow \kappa \geq 1.$

Proof: Fix an interval $J \in \mathcal{D}$ and a weight $w \in A_\infty^d(J; r)$.
For every subinterval $I \in \mathcal{D}(J)$, let the corresponding point in $\mathbb{R}^2$:

$$x^I := \left(\langle w\rangle_I,\ \langle \log w\rangle_I\right) \quad \forall I \in \mathcal{D}(J) \qquad (\text{"Bellman point"})$$



Claim: $x^I \in \Omega_r,\ \forall I \in \mathcal{D}(J)$

Jensen: $x_2^I = \langle \log w\rangle_I \leq \log \langle w\rangle_I = \log x_1^I \Rightarrow x_2^I \leq \log x_1^I$

A_∞ condition: $x_1^I = \langle w\rangle_I \leq r e^{\langle \log w\rangle_I} = r e^{x_2^I} \Rightarrow \log \frac{x_1^I}{r} \leq x_2^I$

Look at any $x^{I_-} = (\langle w\rangle_{I_-}, \langle \log w\rangle_{I_-})$; $x^{I_+} = (\langle w\rangle_{I_+}, \langle \log w\rangle_{I_+}) \Rightarrow x^I = \frac{1}{2}(x^{I_-} + x^{I_+})$
Apply the inequality:

$$B(x^I) \geq \frac{1}{2} B(x^{I_-}) + \frac{1}{2} B(x^{I_+}) + \left(\frac{x_1^{I_+} - x_1^{I_-}}{x_1^I}\right)^2 \qquad \cdot |I|$$

$$\boxed{|I| B(x^I) \geq |I_-| B(x^{I_-}) + |I_+| B(x^{I_+}) + |I| \left(\frac{x_1^{I_+} - x_1^{I_-}}{x_1^I}\right)^2} \quad \forall I \in \mathcal{D}(J)$$

Bellman Induction: Start with J:

$$|J| B(x^J) \geq \underbrace{|J_-| B(x^{J_-})} + \underbrace{|J_+| B(x^{J_+})} + \left(\frac{x_1^{J_+} - x_1^{J_-}}{x_1^J}\right)^2 |J|$$

$$\geq |J_=| B(x^{J_=}) + |J_\mp| B(x^{J_\mp}) + \left(\frac{x_1^{J_\mp} - x_1^{J_=}}{x_1^{J_-}}\right)^2 |J_-|$$

$$\geq |J_\pm| B(x^{J_\pm}) + |J_\ddagger| B(x^{J_\ddagger}) + \left(\frac{x_1^{J_\ddagger} - x_1^{J_\pm}}{x_1^{J_+}}\right)^2 |J_+|$$

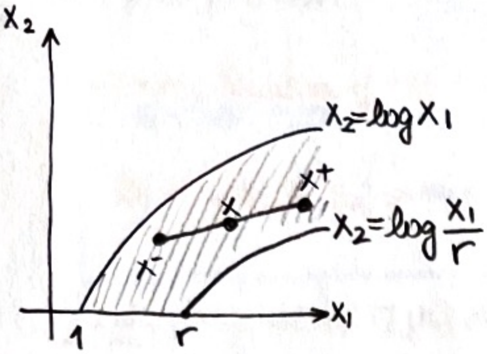
$$\vdots$$

$$\Rightarrow |J| B(x^J) \geq \sum_{I \in \mathcal{D}(J)} \left(\frac{x_1^{I_+} - x_1^{I_-}}{x_1^I}\right)^2 |I| = \sum_{I \in \mathcal{D}(J)} \left(\frac{\langle w\rangle_{I_+} - \langle w\rangle_{I_-}}{\langle w\rangle_I}\right)^2 |I|$$

$\Rightarrow$ Since B is bounded:

$$\sum_{I \in \mathcal{D}(J)} \left(\frac{\langle w\rangle_{I_+} - \langle w\rangle_{I_-}}{\langle w\rangle_I}\right)^2 |I| \leq |J| \underbrace{\|B\|_\infty}_{= C(r)} \qquad \blacksquare$$

Lemma: The function $B(x_1,x_2) := 8(\log x_1 - x_2)$ satisfies the inequality in Theorem 1, on the domain Ω_r.



Proof: Let $x^+, x^- \in \Omega_r$ be such that $x := \frac{1}{2}(x^- + x^+) \in \Omega_r$.

We must show:

$$B(x) \geq \frac{1}{2}\big(B(x^-) + B(x^+)\big) + \left(\frac{x_1^+ - x_1^-}{x_1}\right)^2.$$

Let $\Delta := \frac{1}{2}(x^+ - x^-)$ => if $x = (x_1, x_2)$, we can write:

$$x^+ = (x_1 + \Delta_1,\ x_2 + \Delta_2)$$
$$x^- = (x_1 - \Delta_1,\ x_2 - \Delta_2)$$

$$\Rightarrow B(x) - \frac{1}{2}B(x^-) - \frac{1}{2}B(x^+) - \left(\frac{x_1^+ - x_1^-}{x_1}\right)^2 = 8(\log x_1 - x_2) - \frac{1}{2}\big(8\log(x_1 - \Delta_1) - 8(x_2 - \Delta_2)\big)$$
$$- \frac{1}{2}\big(8\log(x_1 + \Delta_1) - 8(x_2 + \Delta_2)\big) - \left(\frac{2\Delta_1}{x_1}\right)^2$$
$$= 8\log x_1 - 8\cancel{x_2} - 4\log(x_1^2 - \Delta_1^2) + \cancel{4x_2} \cancel{- 4\Delta_2} + \cancel{4x_2} \cancel{+ 4\Delta_2} - 4\left(\frac{\Delta_1}{x_1}\right)^2$$
$$= 4\log\frac{x_1^2}{x_1^2 - \Delta_1^2} - 4\left(\frac{\Delta_1}{x_1}\right)^2 = -4\Bigg(\underbrace{\log\left(1 - \frac{\Delta_1^2}{x_1^2}\right) + \left(\frac{\Delta_1}{x_1}\right)^2}_{<0}\Bigg) \geq 0$$

To see the last inequality, look at the function $f(y) = \log(1 - y^2) + y^2$, for $y \in (0,1)$

$$f'(y) = \frac{-2y}{1-y^2} + 2y = 2y \cdot \frac{-y^2}{1-y^2} < 0$$

y	0	1
$f(y)$	0 $\longrightarrow$	$-\infty$

∎

Remark: $8(\log x_1 - x_2) \leq 8\log r, \ \forall (x_1, x_2) \in \Omega_r$

$$8\log\frac{x_1}{r} \leq 8x_2$$

=> we have Buckley's Inequality with $c(r) = 8\log r$.

II Behind the Curtain: Constructing the Bellman Function

We need to show: $\boxed{\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_I}\right)^2|I| \le C(r)}$ $\forall$ $\boxed{w\in A_\infty^{\mathcal{D}}(J;r)}$

meaning $\mathrm{supp}(w)\subset J$ and $\langle w\rangle_I \le re^{\langle \log w\rangle_I}$, $\forall I\in\mathcal{D}(J)$.

Bellman Function of the Problem:

$$\boxed{\mathbb{B}(x):=\mathbb{B}(x_1,x_2):=\sup\left\{\frac{1}{|J|}\sum_{I\in\mathcal{D}(J)}\left(\frac{\langle w\rangle_{I_+}-\langle w\rangle_{I_-}}{\langle w\rangle_I}\right)^2|I| \;:\; w\in A_\infty^{\mathcal{D}}(J;r),\ \langle w\rangle_J=x_1,\ \langle \log w\rangle_J=x_2\right\}}$$

Remark: Any $w\in A_\infty^{\mathcal{D}}(J;r)$ with $\langle w\rangle_J=x_1$ and $\langle \log w\rangle_J=x_2$ is said to be **admissible** for $\mathbb{B}(x_1,x_2)$.

§ **Properties of $\mathbb{B}$:**

1. **Domain:** $\boxed{\Omega_r:=\left\{x=(x_1,x_2): \log\frac{x_1}{r}\le x_2\le \log x_1\right\}}$

Same as before: Say $w\in A_\infty^{\mathcal{D}}(J;r)$ is admissible for $\mathbb{B}(x_1,x_2)$. Then necessarily

$$\left.\begin{aligned}&\text{Jensen: } x_2=\langle \log w\rangle_J\le \log\langle w\rangle_J=\log x_1\\ &A_\infty\text{ cond.: } x_1=\langle w\rangle_J\le re^{\langle \log w\rangle_J}=re^{x_2}\end{aligned}\right\}\Rightarrow \log\frac{x_1}{r}\le x_2\le \log x_1.$$

2. **$\mathbb{B}$ is independent of the choice of $J\in\mathcal{D}$.**
A linear change of variables maps the set of test functions on some $J\in\mathcal{D}$ to any other dyadic interval $K\in\mathcal{D}$, preserving all averages.
$\Rightarrow$ for any two intervals in $\mathcal{D}$, the supremum in the definition of $\mathbb{B}$ is taken over the same set.
(Very simple observation, but absolutely crucial $\to$ will be the key to obtaining the Main Inequality).

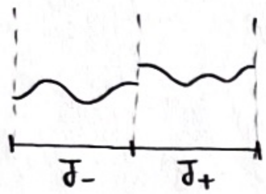
3. Main Inequality: For every pair of points $x^+, x^- \in \Omega_r$ such that their mean $x = \frac{1}{2}(x^+ + x^-)$ is also in Ω_r, the following inequality holds:

$$\boxed{\mathbb{B}(x) \geq \frac{1}{2}\left(\mathbb{B}(x^+) + \mathbb{B}(x^-)\right) + \left(\frac{x_1^+ - x_1^-}{x_1}\right)^2}$$

Proof: The key to the Main Inequality in general is:

Independence of $\mathbb{B}$ wrt the interval allows us to run the Bellman machine on two separate intervals, completely independently of one another.

So: instead of looking at J, we look at J_- and J_+ separately:



Choose weights $w_\pm \in A_\infty^D(J_\pm; r)$ that are admissible for $\mathbb{B}(x^\pm)$, i.e.

$$\langle w_\pm \rangle_{J_\pm} = x_1^\pm \; ; \; \langle \log w_\pm \rangle_{J_\pm} = x_2^\pm,$$

and which "almost" give the supremum in $\mathbb{B}(x^\pm)$, up to some $\varepsilon > 0$:

$$\boxed{\frac{1}{|J_\pm|} \sum_{I \in D(J_\pm)} \left(\frac{\langle w^\pm \rangle_{I_+} - \langle w^\pm \rangle_{I_-}}{\langle w^\pm \rangle_I}\right)^2 |I| \geq \mathbb{B}(x^\pm) - \varepsilon.} \qquad (1)$$

Now concatenate $w_\pm$ into a new weight: $w := w_- \mathbb{1}_{J_-} + w_+ \mathbb{1}_{J_+}$, supported on J.

The assumption that $x = \frac{1}{2}(x^- + x^+) \in \Omega_r$ gives us exactly that $w \in A_\infty^D(J; r)$

$$\left.\begin{aligned} x_1 &:= \langle w \rangle_J = \frac{1}{2}\left(\langle w \rangle_{J_-} + \langle w \rangle_{J_+}\right) = \frac{1}{2}(x_1^- + x_1^+) \\ x_2 &:= \langle \log w \rangle_J = \frac{1}{2}\left(\langle \log w \rangle_{J_-} + \langle \log w \rangle_{J_+}\right) = \frac{1}{2}(x_2^- + x_2^+) \end{aligned}\right\} x = \frac{1}{2}(x_1 + x_2)$$

$\Rightarrow$ **The weight $w \in A_\infty^D(J; r)$ is admissible for $\mathbb{B}(x_1, x_2) = \mathbb{B}(x)$!**

$$\Rightarrow \mathbb{B}(x) \geq \frac{1}{|J|} \sum_{I \in D(J)} \left(\frac{\langle w \rangle_{I_+} - \langle w \rangle_{I_-}}{\langle w \rangle_I}\right)^2 |I|$$

$$= \underbrace{\left(\frac{\langle w \rangle_{J_+} - \langle w \rangle_{J_-}}{\langle w \rangle_J}\right)^2}_{\left(\frac{x_1^+ - x_1^-}{x_1}\right)^2} + \underbrace{\frac{1}{2|J_-|} \sum_{I \in D(J_-)} \left(\frac{\langle w^- \rangle_{I_+} - \langle w^- \rangle_{I_-}}{\langle w^- \rangle_I}\right)^2 |I| + \frac{1}{2|J_+|} \sum_{I \in D(J_+)} \left(\frac{\langle w^+ \rangle_{I_+} - \langle w^+ \rangle_{I_-}}{\langle w^+ \rangle_I}\right)^2 |I|}_{\geq \frac{1}{2}\left(\mathbb{B}(x^-) + \mathbb{B}(x^+)\right) - \varepsilon \text{ by } (1)}$$

$$\Rightarrow \mathbb{B}(x) \geq \frac{1}{2}\left(\mathbb{B}(x^-) + \mathbb{B}(x^+)\right) + \left(\frac{x_1^+ - x_1^-}{x_1}\right)^2 - \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we can take $\lim_{\varepsilon \to 0+}$ and the result follows. ∎